

Hodge cohomology of invertible sheaves

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1 Introduction

Let k be an algebraically closed field and let X/k be a smooth projective connected k -scheme. Let L be an invertible sheaf on X , and for each integer m , let

$$H_{Hdg}^m(X/k, L) := \bigoplus_{a+b=m} H^b(X, L \otimes \Omega_{X/k}^a).$$

We wish to study how the dimensions of the k -vector spaces $H_{Hdg}^m(X/k, L)$ and $H^b(X, L \otimes \Omega_{X/k}^a)$ vary with L . For example, if k has characteristic zero, Green and Lazarsfeld [4] proved that for given i, j, m , the subloci

$$\{L \in \text{Pic}^0(X) : \dim H^i(X, \Omega_X^j \otimes L) \geq m\}$$

of $\text{Pic}^0(X)$ are translates of abelian subvarieties, and Simpson [12] showed that they in fact are translates by torsions points. Both these papers use analytic methods, but Pink and Roessler [10] obtained the same results purely algebraically, using the technique of mod p reduction and the decomposition theorem of Deligne-Illusie. A key point of their proof is the fact that if $L^n \cong \mathcal{O}_X$ for some positive integer n , then for all natural numbers a with $(a, n) = 1$ one has

$$\dim H_{Hdg}^m(X/k, L) = \dim H_{Hdg}^m(X/k, L^a) \quad (1)$$

([10, Proposition 3.5]). They conjecture that equation 1 remains true in characteristic $p > 0$ if X/k lifts to $W_2(k)$ and has dimension $\leq p$. The purpose of this note is to discuss a few aspects of this conjecture and some variants.

Our main result (see Theorem 7) says that the conjecture is true if $n = p$ and X is ordinary in the sense of Bloch-Kato [2, Definition 7.2]. We also explain in section 2 some motivic variants of (1) and, in particular in Proposition 1, a proof (due to Pink and Roessler) of the characteristic zero case of (1), using the language of Grothendieck Chow motives. See [7, 9.3] for a discussion of a related problem using similar techniques. We should remark that there are also some log versions of these questions, which we will not make explicit.

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2 Motivic variants

Question 1 *Let X be a smooth projective connected variety defined over an algebraically closed field k . Let L be an invertible sheaf on X and n a positive integer such that $L^n \cong \mathcal{O}_X$. Is*

$$\dim H_{Hdg}^m(X/k, L^i) = \dim H_{Hdg}^m(X/k, L)$$

for every i relatively prime to n ?

Let us explain how this question can be given a motivic interpretation. We refer to [11] for the definition of Grothendieck's Chow motives over a field k . In particular, objects are triples (Y, p, n) where Y is a smooth projective variety over k , p is an element $CH^{\dim(Y)}(Y \times_k Y) \otimes \mathbb{Q}$, the rational Chow group of $\dim(Y)$ -cycles, which, as a correspondence, is an idempotent, and n is a natural number.

Let $\pi : Y \rightarrow X$ be a principal bundle under a k -group scheme μ , where X and Y are smooth and projective over k . Recall that this means that there is a k -group scheme action $\mu \times_k Y \rightarrow Y$ with the property that one has an isomorphism

$$(\xi, y) \mapsto (y, \xi y) : \mu \times_k Y \cong Y \times_X Y \subseteq Y \times_k Y.$$

Thus a point $\xi \in \mu(k)$ defines a closed subset Γ_ξ of $Y \times_k Y$, the graph of the endomorphism of Y defined by ξ . The map $\xi \mapsto \Gamma_\xi$ extends uniquely to a

map of \mathbb{Q} -vector spaces

$$\Gamma : \mathbb{Q}[\mu(k)] \rightarrow CH^{\dim(Y)}(Y \times_k Y) \otimes \mathbb{Q}.$$

Here $\mathbb{Q}[\mu(k)]$ is the \mathbb{Q} -group algebra, so the product structure is induced by the product of k -roots of unity. We can think of $CH^{\dim(Y)}(Y \times_k Y) \otimes \mathbb{Q}$ as a \mathbb{Q} -algebra of correspondences acting on $CH^*(Y) \otimes \mathbb{Q}$, where for $\beta \in CH^s(Y) \otimes \mathbb{Q}$, $\gamma \in CH^{\dim(Y)}(Y \times_k Y) \otimes \mathbb{Q}$, one defines as usual

$$\gamma \cdot \beta := (p_2)_*(\gamma \cup p_1^* \beta).$$

Then the map Γ is easily seen to be compatible with composition, as on closed points $y \in Y$ one has $\Gamma_\xi(y) = \xi \cdot y$. In particular if $\xi \in \mathbb{Q}[\mu]$ is idempotent in the group ring $\mathbb{Q}[\mu(k)]$, then $\Gamma_\xi \cong Y \times \xi$ is idempotent as a correspondence. In this case we let Y_ξ be the Grothendieck Chow motive $(Y, \xi, 0)$.

Let L be an n -torsion invertible sheaf on smooth irreducible projective scheme X/k . Recall that the choice of an \mathcal{O}_X -isomorphism $L^n \xrightarrow{\alpha} \mathcal{O}_X$ defines an \mathcal{O}_X -algebra structure on

$$\mathcal{A} := \bigoplus_{i=0}^{n-1} L^i \quad (2)$$

via the tensor product $L^i \times L^j \rightarrow L^i \otimes_{\mathcal{O}_X} L^j = L^{i+j}$ for $i+j < n$ and its composition with the isomorphism $L^i \times L^j \rightarrow L^i \otimes_{\mathcal{O}_X} L^j = L^{i+j} \xrightarrow{\alpha^{-1}} L^{i+j-n}$ for $0 \leq i+j-n$. Then the corresponding X -scheme $\pi : Y := \text{Spec}_X \mathcal{A} \rightarrow X$ is a torsor under the group scheme μ_n of n th roots of unity. Indeed, locally Zariski on X , $\mathcal{A} \cong \mathcal{O}_X[t]/(t^n - u)$ for a local unit u , the μ_n -action is defined by $\mathcal{A} \rightarrow \mathcal{A} \otimes \mathbb{Q}[\zeta]/(\zeta^n - 1)$, $t \mapsto t\zeta$, and the torsor structure is given by $\mathcal{A} \otimes \mathbb{Q}[\zeta]/(\zeta^n - 1) \cong \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}$, $(t, \zeta) \mapsto (t, t\zeta)$. This construction defines an equivalence between the category of pairs (L, α) and the category of μ_n -torsors over X . Assuming now that n is invertible in k , μ_n is étale, hence π is étale and Y is smooth and projective over k . Note the character group $X_n := \text{Hom}(\mu_n, \mathbf{G}_m)$ is cyclic of order n with a canonical generator (namely, the inclusion $\mu_n \rightarrow \mathbf{G}_m$). By construction, the direct sum decomposition (2) of \mathcal{A} corresponds exactly to its eigenspace decomposition according to the characters of μ_n .

We can now apply the general construction of motives to this situation. Since μ_n is étale over the algebraically closed field k , it is completely determined by the finite group $\Gamma := \mu_n(k)$, which is cyclic of order n . The group

algebra $\mathbb{Q}[\Gamma]$ is a finite separable algebra over \mathbb{Q} , hence is a product of fields:

$$\mathbb{Q}[\Gamma] = \prod E_e.$$

Here $E_e = \mathbb{Q}[T]/(\Phi_e(T)) = \mathbb{Q}(\xi_e)$, where e is a divisor of n , $\Phi_e(T)$ is the cyclotomic polynomial, and ξ_e is a primitive e th root of unity. There is an (indecomposable) idempotent e corresponding to each of these fields, and for each e we find a Chow motive Y_e .

The indecomposable idempotents of $\mathbb{Q}[\Gamma]$ can also be thought of as points of the spectrum T of $\mathbb{Q}[\Gamma]$. If K is a sufficiently large extension of \mathbb{Q} , then

$$T(K) = \text{Hom}_{\text{Alg}}(\mathbb{Q}[\Gamma], K) = \text{Hom}_{\text{Gr}}(\Gamma, K^*), \quad (3)$$

$$\text{and } K \otimes \mathbb{Q}[\Gamma] \cong K[\Gamma] \cong K^{T(K)}. \quad (4)$$

Thus $T(K)$ can be identified with the character group X_n of Γ , and is canonically isomorphic to $\mathbf{Z}/n\mathbf{Z}$, with canonical generator the inclusion $\Gamma \subseteq k$. Suppose that K/\mathbb{Q} is Galois. Then $\text{Gal}(K/\mathbb{Q})$ acts on $T(K)$, and the points of T correspond to the $\text{Gal}(K/\mathbb{Q})$ -orbits. By the theory of cyclotomic extensions of \mathbb{Q} , this action factors through a surjective map

$$\text{Gal}(K/\mathbb{Q}) \rightarrow (\mathbf{Z}/n\mathbf{Z})^*$$

and the usual action of $(\mathbf{Z}/n\mathbf{Z})^*$ on $\mathbf{Z}/n\mathbf{Z}$ by multiplication. Thus the orbits correspond precisely to the divisors d of n ; we shall associate to each orbit S the index d of the subgroup of $\mathbf{Z}/n\mathbf{Z}$ generated by any element of S . (Note that in fact the image of d in $\mathbf{Z}/n\mathbf{Z}$ belongs to S .) We shall thus identify the indecomposable idempotents of $\mathbb{Q}[\Gamma]$ and the divisors of n .

Let us suppose that $k = \mathbf{C}$. Then we can consider the Betti cohomologies of X and Y , and in particular the group algebra $\mathbb{Q}[\Gamma]$ operates on $H^m(Y, \mathbb{Q})$. We can thus view $H^m(Y, \mathbb{Q})$ as a $\mathbb{Q}[\Gamma]$ -module, which corresponds to a coherent sheaf $\tilde{H}^m(Y, \mathbb{Q})$ on T . If e is an idempotent of $\mathbb{Q}[\Gamma]$, then $H^m(Y_e, \mathbb{Q})$ is the image of the action of e on $H^m(Y, \mathbb{Q})$, or equivalently, it is the stalk of the sheaf $\tilde{H}^m(Y, \mathbb{Q})$ at the point of T corresponding to e , or equivalently, it is $H^m(Y, \mathbb{Q}) \otimes E_e$ where the tensor product is taken over $\mathbb{Q}[\Gamma]$. If K is a sufficiently large field as above, then equation (4) induces an isomorphism of K -vectors spaces:

$$H^m(Y_e, \mathbb{Q}) \otimes_{\mathbb{Q}} K \cong \bigoplus \{H^m(Y, K)_t : t \in T^e(K)\},$$

where here $T^e(K)$ means the set of points of $T(K)$ in the Galois orbit corresponding to e , and $H^m(Y, K)_t$ means the t -eigenspace of the action of Γ on $H^m(Y, \mathbb{Q}) \otimes_{\mathbb{Q}} K$. The de Rham and Hodge cohomologies of Y_e are defined in the same way: they are the images of the actions of the idempotent e acting on the k -vector spaces $H_{DR}(Y/k)$ and $H_{Hdg}(Y/k)$.

The following result is due to Pink and Roessler. Their article [10] contains a proof using reduction modulo p techniques and the results of [3]; the following analytic argument is based on oral communications with them.

Proposition 1 *The answer to question 1 is affirmative if k is a field of characteristic zero.*

Proof: As both sides of the equality in Question 1 satisfy base change with respect to field extensions, we may assume that $k = \mathbf{C}$. Let $i \rightarrow t_i$ denote the isomorphism $\mathbf{Z}/n\mathbf{Z} \cong T(\mathbf{C})$. For each divisor e of n there is a corresponding idempotent e of $\mathbb{Q}[\Gamma] \subseteq K[\Gamma]$, the sum over all i such that $t_i \in T^e(\mathbf{C})$. Consider the Hodge cohomology of the motive Y_e :

$$\begin{aligned} H_{Hdg}^m(Y_e/\mathbf{C}) &:= H_{Hdg}^m(Y/\mathbf{C}) \otimes_{\mathbb{Q}[\Gamma]} E_e \cong H_{Hdg}^m(Y/\mathbf{C}) \otimes_{\mathbf{C}[\Gamma]} (\mathbf{C} \otimes E_e). \\ &\cong \bigoplus \{H_{Hdg}^m(Y/\mathbf{C})_i : i \in T^e(k)\}. \end{aligned}$$

Since $\pi: Y \rightarrow X$ is finite and étale,

$$\begin{aligned} H^b(Y, \Omega_{Y/\mathbf{C}}^a) &\cong H^b(X, \pi_* \pi^* \Omega_{X/\mathbf{C}}^a) \cong H^b(X, \Omega_{X/\mathbf{C}}^a \otimes \pi_* \mathcal{O}_Y) \\ &\cong \bigoplus \{H^b(X, \Omega_{X/\mathbf{C}}^a \otimes L^i) : i \in \mathbf{Z}/n\mathbf{Z}\}. \end{aligned}$$

Thus

$$H_{Hdg}^m(Y/\mathbf{C}) \cong \bigoplus \{H_{Hdg}^m(X, L^i) : i \in \mathbf{Z}/n\mathbf{Z}\},$$

and hence from the explicit description of the action of μ_n on \mathcal{A} above it follows that

$$H_{Hdg}^m(Y_e/\mathbf{C}) = \bigoplus \{H_{Hdg}^m(X, L^i) : i \in T_e(\mathbf{C})\}.$$

The Hodge decomposition theorem for Y provides us with an isomorphism:

$$H_{Hdg}^m(Y/\mathbf{C}) \cong \mathbf{C} \otimes H^m(Y, \mathbb{Q}),$$

compatible with the action of $\mathbb{Q}[\Gamma]$. This gives us, for each idempotent e , an isomorphism of $\mathbf{C} \otimes E_e$ -modules.

$$H_{Hdg}^m(Y_e/\mathbf{C}) \cong \mathbf{C} \otimes H^m(Y_e, \mathbb{Q}).$$

The action on $\mathbf{C} \otimes H^m(Y_e, \mathbb{Q})$ on the right just comes from the action of E_e on $H^m(Y_e, \mathbb{Q})$ by extension of scalars. Since E_e is a field, $H^m(Y_e, \mathbb{Q})$ is free as an E_e -module, and hence the $\mathbf{C} \otimes E_e$ -module $H_{Hdg}^m(Y_e/\mathbf{C})$ is also free. It follows that its rank is the same at all the points $t \in T^e(\mathbf{C})$, affirming Question 1. \square

Let us now formulate an analog of Question 1 for the ℓ -adic and crystalline realizations of the motive Y_e in characteristic p .

Question 2 *Suppose that k is an algebraically closed field of characteristic p and $(n, p) = 1$. Let ℓ be a prime different from p , let e be a divisor of n , and let E_e be the corresponding factor of $\mathbb{Q}[\Gamma]$. Is it true that each $H^m(Y_e, \mathbb{Q}_\ell)$ is a free $\mathbb{Q}_\ell \otimes E_e$ -module? And is it true that $H_{cris}^m(Y_e/W) \otimes \mathbb{Q}$ is a free $W \otimes E_e$ -module, where $W := W(k)$?*

If K is an extension of \mathbb{Q}_ℓ (resp. of $W(k)$) which contains a primitive n th root of unity, then as above we have a eigenspace decompositions:

$$K \otimes H^m(Y_{\acute{e}t}, \mathbb{Q}_\ell) \cong \bigoplus \{H^m(Y_{\acute{e}t}, K)_t : t \in T(K)\} \quad (5)$$

$$K \otimes H^m(Y_{cris}/W(k)) \cong \bigoplus \{H^m(Y_{cris}, K)_t : t \in T(K)\}, \quad (6)$$

and this question asks whether the K -dimension of the t -eigenspace is constant over the orbits $T_e(K) \subseteq T(K)$.

We show in the sequel that the question has a positive answer.

Suppose first that X/k lifts to characteristic zero, *i.e.*, that there exists a complete discrete valuation ring V with residue field k and fraction field of characteristic zero and a smooth proper \tilde{X}/V whose special fiber is X/k . Let X_m be the closed subscheme of \tilde{X} defined by π^{m+1} , where π is a uniformizing parameter of V . Choose a trivialization α of L^n . It follows from Theorem 18.1.2 of [6] that the étale μ_n -torsor Y on X corresponding to (L, α) lifts to X_m , uniquely up to a unique isomorphism, and hence that the same is true for (L, α) . This fact can also be seen by chasing the exact sequences of cohomology corresponding to the commutative diagram of exact sequences

in the étale topology

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & (7) \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}_X & \xrightarrow{n \cong} & \mathcal{O}_X & & \\
 & & \downarrow a \mapsto 1 + \pi^m a & & \downarrow a \mapsto 1 + \pi^m a & & \\
 1 & \longrightarrow & \mu_n & \longrightarrow & \mathcal{O}_{X_m}^\times & \xrightarrow{n} & \mathcal{O}_{X_m}^\times \longrightarrow 1 \\
 & & \downarrow = & & \downarrow & & \\
 1 & \longrightarrow & \mu_n & \longrightarrow & \mathcal{O}_{X_{m-1}}^\times & \xrightarrow{n} & \mathcal{O}_{X_{m-1}}^\times \longrightarrow 1 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 1 & & 1
 \end{array}$$

By Grothendieck's fundamental theorem for proper morphisms, it follows that (L, α) and Y lift to $(\tilde{L}, \tilde{\alpha})$ and \tilde{Y} on \tilde{X} . Then by the étale to Betti and Betti to crystalline comparison theorems, we see that under the lifting assumption, the answer to Question 2 is affirmative.

In fact, the lifting hypothesis is superfluous, but this takes a bit more work.

Claim 2 *The answer to Question 2 is affirmative.*

Proof: It is trivially true that $H^m(Y_e, \mathbb{Q}_\ell)$ is free over $\mathbb{Q}_\ell \otimes E_e$ if $\mathbb{Q}_\ell \otimes E_e$ is a field. If $(\ell, n) = 1$, this is the case if and only if $(\mathbf{Z}/e\mathbf{Z})^*$ is cyclic and generated by ℓ . More generally, assuming ℓ is relatively prime to n , there is a decomposition of $\mathbb{Q}_\ell[\Gamma]$ into a product of fields $\mathbb{Q}_\ell[\Gamma] \cong \prod E_{\ell,e}$, where now e ranges over the orbits of $\mathbf{Z}/n\mathbf{Z}$ under the action of the cyclic subgroup of $(\mathbf{Z}/n\mathbf{Z})^*$ generated by ℓ . This is indeed the unramified lift of the decomposition of $A = \mathbb{F}_\ell[\Gamma]$ into a product of finite extensions of \mathbb{F}_ℓ , corresponding to the orbits of Frobenius on the geometric points of A . This shows at least that the dimension of $H^m(Y, K)_t$ in (5) is, as a function of t , constant over the ℓ -orbits.

For the general statement, let K be an algebraically closed field containing \mathbb{Q}_ℓ for all primes $\ell \neq p$, and containing $W(k)$. For $\ell \neq p$ let $V_\ell := H^m(Y_{\text{ét}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} K$, and let $V_p := H^m(Y_{\text{cris}}, W(k)) \otimes_{W(k)} K$. Then each V_ℓ is a finite-dimensional representation of Γ , and the isomorphisms (5)

and (6) are just its decomposition as a direct sum of irreducible representations:

$$V_\ell \cong \bigoplus \{n_{\ell,i} V_i : i \in \mathbf{Z}/n\mathbf{Z}\},$$

where $V_i = K$, with $\gamma \in \Gamma$ acting by multiplication by γ^i . By [8, Theorem 2.2)] (and [1], [5] and [9] for the existence of cycle classes in crystalline cohomology) the trace of any $\gamma \in \Gamma$ acting on V_ℓ is an integer independent of ℓ , including $\ell = p$. Since Γ is a finite group, it follows from the independence of characters that for each i , $n_i := n_{\ell,i}$ is independent of ℓ . We saw above that $n_{\ell,\ell i} = n_{\ell,i}$ if $(\ell, n) = 1$ and $\ell \neq p$, so that in fact $n_{\ell i} = n_i$ for all $\ell \neq p$ with $(\ell, n) = 1$. Since the group $(\mathbf{Z}/n\mathbf{Z})^*$ is generated by all such ℓ , it follows that n_i is indeed constant over the ℓ -orbits. \square

What does this tell us about Question 1? If $(p, n) = 1$ and k is algebraically closed, $W[\Gamma]$ is still semisimple, and can be written canonically as a product of copies of W , indexed by $i \in T(W) \cong \mathbf{Z}/n\mathbf{Z}$. For every $t \in T(W) \cong T(k)$, we have an injective base change map from crystalline to de Rham cohomology: $k \otimes H^m(Y/W)_t \rightarrow H^m(Y/k)_t$.

Question 3 *In the above situation, is $H^q(Y/W)$ torsion free when $(p, n) = 1$?*

If the answer is yes, then the maps $k \otimes H^m(Y/W)_t \rightarrow H^m(Y/k)_t$ are isomorphisms, and this means that we can compute the dimensions of the de Rham eigenspaces from the ℓ -adic ones. Assuming also that the Hodge to de Rham spectral sequence of Y/k degenerates at E_1 , this should give an affirmative answer to Question 1. Note that if X/k lifts mod p^2 , then Y/k lifts mod p^2 as well, and if the dimension is less than or equal to p , the E_1 -degeneration is true by [3].

Of course, there is no reason for Question 3 to have an affirmative answer in general. Is there a reasonable hypothesis on X which guarantees it? For example, is it true if the crystalline cohomology of X/W is torsion free?

3 The p -torsion case in characteristic p

Let us assume from now on that k is a perfect field of characteristic $p > 0$. In this case we can reduce question 1 to a question about connections, using the following construction of [3]. First let us recall some standard notations.

Let X' be the pull back of X via the Frobenius of k , let $\pi: X' \rightarrow X$ be the projection, and let $F: X \rightarrow X'$ and $F_X: X \rightarrow X$ be the relative and absolute Frobenius morphisms. Then $F_X^* L = L^p = F^* L'$, where $L' := \pi^* L$. Then $L^p = F^{-1} L' \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X$ is endowed with the Frobenius descent connection $1 \otimes d$, *i.e.* the unique connection spanned by its flat sections L' . In general, for a given integrable connection (E, ∇) , we set

$$H_{DR}^i(X, (E, \nabla)) = \mathbb{H}^i(X/k, (\Omega_{X/k}^\bullet \otimes E, \nabla)),$$

and we use again the notation

$$H_{Hdg}^i(X/k, L) = \bigoplus_{a+b=i} H^b(X, \Omega_{X/k}^a \otimes L)$$

and write h_{DR}^i and h_{Hdg}^i for the respective dimensions of these spaces.

Proposition 3 *Let L be an invertible sheaf on a smooth proper scheme X over k and let ∇ be the Frobenius descent connection on L^p . Suppose that X/k lifts to $W_2(k)$ and has dimension at most p . Then for every natural number m ,*

$$h_{DR}^m(X/k, (L^p, \nabla)) = h_{Hdg}^m(X/k, L).$$

Corollary 4 *Under the assumptions of Proposition 3, if $L^p \cong \mathcal{O}_X$ and $\omega := \nabla(1)$, then for any integer a ,*

$$h_{Hdg}^m(X/k, L^a) = h_{DR}^m(X/k, (\mathcal{O}_X, d + a\omega)).$$

Remark 5 If p divides a , this just means the degeneration of the Hodge to de Rham spectral sequence for (\mathcal{O}_X, d) .

Proof: Let $Hdg_{X'/k}^\bullet$ denote the Hodge complex of X'/k , *i.e.*, the direct sum $\bigoplus_i \Omega_{X'/k}^i[-i]$. Recall from [3] that the lifting yields an isomorphism in the bounded derived category of $\mathcal{O}_{X'}$ -modules:

$$Hdg_{X'/k}^\bullet \cong F_*(\Omega_{X/k}^\bullet, d).$$

Tensoring this isomorphism with $L' := \pi^* L$ and using the projection formula for F , we find an isomorphism

$$Hdg_{X'/k}^\bullet \otimes L' \cong F_*(\Omega_{X/k}^\bullet \otimes L^p, \nabla).$$

Hence

$$H_{Hdg}^m(X/k, L) \xrightarrow{F_k^* \cong} H_{Hdg}^m(X'/k, L') \xleftarrow{F_* \cong} H_{DR}^m(X, (L^p, \nabla)).$$

This proves the proposition. If $L^p = \mathcal{O}_X$, the corresponding Frobenius descent connection ∇ on \mathcal{O}_X is determined by $\omega_L := \nabla(1)$. It follows from the tensor product rule for connections that $\omega_{L^a} = a\omega_L$ for any integer a . \square

The corollary suggests the following question.

Question 4 *Let ω be a closed one-form on X and let c be a unit of k . Is the dimension of $H_{DR}^m(X, (\mathcal{O}_X, d + c\omega))$ independent of c ?*

Remark 6 Some properness is necessary, since the p -curvature of $d_\omega := d + \omega$ can change from zero to non-zero as one multiplies by an invertible constant. If the p -curvature is non-zero, then the sheaf $\mathcal{H}^0(\Omega_{X/k}, d_\omega)$ vanishes, and hence so does $H^0(X, (\Omega_{X/k}, d_\omega))$. If the p -curvature vanishes, then $\mathcal{H}^0(\Omega_{X/k}, d_\omega)$ is an invertible sheaf L , which can have nontrivial sections if X is allowed to shrink. However, since by definition, $L \subset \mathcal{O}_X$, it can have a global section on a proper X only if $L = \mathcal{O}_X$.

We can answer Question 4 under a strong hypothesis.

Theorem 7 *Suppose that X/k is smooth, proper, and ordinary in the sense of Bloch and Kato [2, Definition 7.2]: $H^i(X, B_{X/k}^j) = 0$ for all i, j , where*

$$B_{X/k}^j := \text{Im} \left(d: \Omega_{X/k}^{j-1} \rightarrow \Omega_{X/k}^j \right).$$

Then the answer to question 4 is affirmative. Hence if X/k lifts to $W_2(k)$, has dimension at most p , and if $n = p$, the answer to Question 1 is also affirmative.

We begin with the following lemmas.

Lemma 8 *Let ω be a closed one-form on X , and let*

$$d_\omega := d + \omega \wedge \quad : \quad \Omega_{X/k}^\bullet \rightarrow \Omega_{X/k}^{\bullet+1}.$$

Then the standard exterior derivative induces a morphism of complexes:

$$(\Omega_{X/k}^\bullet, d_\omega) \xrightarrow{\delta} (\Omega_{X/k}^\bullet, d_\omega)[1].$$

Proof: If α is a section of $\Omega_{X/k}^q$,

$$\begin{aligned} dd_\omega(\alpha) &= d(d\alpha + \omega \wedge \alpha) \\ &= dd\alpha + d\omega \wedge \alpha - \omega \wedge d\alpha \\ &= -\omega \wedge d\alpha. \end{aligned}$$

Since the sign of the differential of the complex $(\Omega_{X/k}^\bullet, d_\omega)[1]$ is the negative of the sign of the differential of $(\Omega_{X/k}^\bullet, d_\omega)$,

$$\begin{aligned} d_\omega d(\alpha) &= -(d + \omega \wedge)(d\alpha) \\ &= -\omega \wedge d\alpha \end{aligned}$$

□

Lemma 9 *Let $Z^\bullet := \ker(d) \subseteq (\Omega_{X/k}^\bullet, d_\omega)$ and $B^\bullet := \text{Im}(d)[-1] \subseteq (\Omega_{X/k}^\bullet, d_\omega)$. Then for any $a \in k^*$, multiplication by a^i in degree i induces isomorphisms*

$$\begin{aligned} (Z^\bullet, d_\omega) &\xrightarrow{\lambda_a} (Z^\bullet, d_{a\omega}) \\ (B^\bullet, d_\omega) &\xrightarrow{\lambda_a} (B^\bullet, d_{a\omega}). \end{aligned}$$

Proof: It is clear that the boundary map d_ω on Z^\bullet and on B^\bullet is just wedge product with ω . □

Proof of Theorem 7 The morphism δ of Lemma 8 induces an exact sequence:

$$0 \rightarrow (Z^\bullet, d_\omega) \rightarrow (\Omega_{X/k}^\bullet, d_\omega) \xrightarrow{\delta} (B^\bullet, d_\omega)[1] \rightarrow 0. \quad (8)$$

As X/k is ordinary, the E_1 term of the first spectral sequence for (B^\bullet, d_ω) is $E_1^{i,j} = H^j(X, B^i) = 0$, and it follows that the hypercohomology of (B^\bullet, d_ω) vanishes, for every ω . Hence the natural map $H^q(Z^\bullet, d_\omega) \rightarrow H^q(\Omega_{X/k}^\bullet, d_\omega)$ is an isomorphism. Since the dimension of $H^q(Z^\bullet, d_\omega)$ is unchanged when ω is multiplied by a unit of k , the same is true of $H^q(\Omega_{X/k}^\bullet, d_\omega)$. This completes the proof of Theorem 7. □

Remark 10 A simple Riemann-Roch computation shows that on curves, question 1 has a positive answer with no additional assumptions. Indeed, if L is a nontrivial torsion sheaf, then its degree is zero and it has no global sections. It follows that $h^1(L) = g - 1$. Since the same is true for L^{-1} , $h^0(L \otimes \Omega_X^1) = h^1(L^{-1}) = g - 1$, and $h^1(L \otimes \Omega_X^1) = h^0(L^{-1}) = 0$.

Remark 11 In the absence of the ordinarity hypothesis, one can ask if the rank of the boundary map

$$\partial_\omega: H^{q+1}(B^\bullet, \omega \wedge) \rightarrow H^{q+1}(Z^\bullet, \omega \wedge)$$

of (8) changes if ω is multiplied by a unit of k . To analyze this question, let

$$c_\omega: (B^\bullet, \omega \wedge) \rightarrow (Z^\bullet, \omega \wedge)$$

be the morphism in the derived category $D(X', \mathcal{O}_{X'})$ defined by the exact sequence (8), so that ∂_ω can be identified with $H^{q-1}(c_\omega)$. Similarly, the exact sequence

$$0 \rightarrow (Z^\bullet, \omega \wedge) \rightarrow (\Omega^\bullet, \omega \wedge) \rightarrow (B^\bullet, \omega \wedge)[1] \rightarrow 0$$

defines a morphism

$$a_\omega: (B^\bullet, \omega \wedge) \rightarrow (Z^\bullet, \omega \wedge)$$

in $D(X', \mathcal{O}_{X'})$ as well. There is also an inclusion morphism:

$$b_\omega: (B^\bullet, \omega \wedge) \rightarrow (Z^\bullet, \omega \wedge).$$

Then it is not difficult to check that $c_\omega = a_\omega + b_\omega$. If $a \in k^*$, we have isomorphisms of complexes

$$\begin{aligned} \lambda_a: (Z^\bullet, \omega \wedge) &\rightarrow (Z^\bullet, a\omega \wedge) \\ \lambda_a: (B^\bullet, \omega \wedge) &\rightarrow (B^\bullet, a\omega \wedge) \end{aligned}$$

Using these as identifications, one can check that $c_{a\omega} = a^{-1}a_\omega + b_\omega$. This would suggest a negative answer to Question 4, but we do not have an example.

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